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**PROFESSOR:** All right. So last time we talked about methods for solving recurrences, and we spent most of our time talking about divide-and-conquer recurrences. These are recurrences where you break the problem up into much smaller sub-problems, like half the size or  $2/3$  the size. And they come up quite a bit in computer science when you're doing algorithm design and algorithm analysis.

Today, we're going to spend our time talking about a different kind of recurrence that's called a linear recurrence. They also come up in computer science and a lot of other fields. Now, I'm going to give you the formal definition of a linear recurrence later. First I want to start with an example, and this is an example of a linear recurrence that comes up in population modeling. In fact, it comes up in a lot of places.

And to start it off, we're going to analyze a particular problem that we call the graduate student job problem or the graduate student job nightmare. And the question here is, will your TAs be able to get a job as a professor somewhere when they get their PhD? So this is a problem they worry about a lot.

And the idea in this problem is that there's some discipline, say computer science. And if you look in all the universities in the world, there's  $M$  faculty lines. So the total number of jobs is  $M$ , and that's a fixed value. Because of budgetary constraints, universities aren't growing, and so they're not going to create more computer science faculty positions. It's going to stay fixed over time. So it's a constant.

And every year, professors generate more graduates who become professors. And in particular, we're going to assume that in this field every professor graduates 1 student who goes on to become a professor, or tries to if there's jobs. So each professor generates 1 graduate who becomes a new professor, as long as there's jobs, per year.

With one exception, and that's first-year professors because they're too busy learning how to teach, getting grants, doing administrative stuff, just figuring out how it all works. So they don't

have time to produce any grad students. So first year or new professors don't produce anything. Except first-year profs-- oops, let me correct that. So first-year profs have 0.

Now, matters are made worse by the fact that Congress passed a law, and this is true, that more or less bars mandatory retirements in colleges. And so that means there are no retirements in this problem, and we're going to assume the faculty live forever. And so once you fill a position, it's filled forever. It doesn't exist anymore. In fact, if you walk around the math department, you can see the impact of this.

[LAUGHTER]

I think the median age is now well into the 70s. And there will be actually a phenomenon over the next 10 years as the math faculty progress into their 80s where they actually, in reality, probably do start to retire. And there will be a whole new wave of people hired in mathematics, just as happened back in the '50s and '60s after the Sputnik crisis where a lot of mathematicians were hired. And then they stayed in those positions, for our purposes, forever.

Now, the question is, when do all the jobs get filled? So when are all M jobs filled by this process? Now, to be able to answer this question, we need one more piece of information. Can anybody think about one more fact here that we need before we start going off and answering this question, analyzing it? Yeah?

**AUDIENCE:** How many professors do we start with?

**PROFESSOR:** How many professors do we start with? What's the boundary condition? What's the base case if we were doing an induction? All right. So let's say the boundary condition, and this always is important with recurrences, is that the first professor is hired in year 1, and there were none before that person. All right? So in year 1 there's 1 professor, and that professor's new.

OK. So now we have all the information necessary to solve the problem. So let's do that and set up a recurrence. We're going to define  $f$  of  $n$  to be the number of professors during year  $n$ . And we know from our boundary condition that in year 0 there were none, and in year 1 there was 1, a new one. What is  $f$  of 2? How many professors are there in year 2?

**AUDIENCE:** 1.

**PROFESSOR:** 1, because the one that was there was too young to do anything. So he or she is the only one left, all right, in year 2. What is  $f$  of 3? How many profs are there in year 3? 2-- the one you

had, and by that point that one is old enough to produce a new one. All right. What's  $f$  of 4?

**AUDIENCE:** 3.

**PROFESSOR:** 3, the 2 you had, and there was one who's been there 2 years to produce a new one. All right. And let's do one more,  $f$  of 5?

**AUDIENCE:** 5.

**PROFESSOR:** 5. These guys produced 1 each, and you had 3 existing. All right. So we can actually write down the recurrence now by sort of the process we just went through. For years 2 and beyond, the number of professors in year  $n$  is the number that we had last year-- that's the previous ones-- plus the number that were generated, new professors, the graduate students that graduated. In terms of  $f$ , how many new ones are there?

**AUDIENCE:**  $f$  of  $n$  minus 2.

**PROFESSOR:**  $f$  of  $n$  minus 2, because that's the number of professors that were there a couple years ago, and they are now generating them. So that's the new ones there. Do people recognize that recurrence? Yeah, it's pretty famous. How many people have not seen? This is called the Fibonacci recurrence. It produces the Fibonacci numbers. How many people have not seen it? Yeah, very famous. Yeah, pretty much everybody has seen that.

Actually, this is the first recurrence that was known to be studied of all recurrences. It was published by Leonardo Fibonacci of Pisa in 1202, all right, so over 800 years ago, and he studied it for modeling the population growth of rabbits. And the idea is that you have a pair of rabbits, and in every month after their first year of life-- sorry, first month of life-- they produce two new rabbits.

All right? So it's the same notion as here. The first pair you do nothing, but after that you're reproducing one for one. And that's an abstraction, but it produces the same recurrence.

Now, Fibonacci is credited with discovering it. Really that means he's the one that told the Europeans about it back then. And in fact, it's now been traced back in to about 200 BC. The Indian mathematicians knew all about Fibonacci's recurrence, and they were using it to study certain properties of grammar and music way back at 200 BC.

This recurrence comes up in all sorts of applications. Kepler used it in the 16th century while

studying how the leaves of a flower are arranged around the stem-- how many leaves you have in sort of each level coming out around a stem. The first solution was discovered by de Moivre in the 18th century. And we're going to talk about how to solve this in a minute. But he was the first one to figure out a closed form expression for  $f$  of  $n$ .

Lame used it in the 19th century when he was studying the Euclidean GCD algorithm. You know that pulverizer thing and doing GCDs? It turns out that if you want to analyze the running time, well, you get a Fibonacci recurrence comes into play there, and that was discovered in the 19th century.

In the 20th century, it was used in the study of optics, economics, and algorithms, and it was named for Fibonacci. It got a name in the 19th century. In fact, this is so popular and used in so many places there is a journal in mathematics called the *Fibonacci Quarterly Journal* where they study these kinds of things.

So today what we're going to do is solve this recurrence and actually solve a much broader family of recurrences called linear recurrences. And we're going to get a closed form. I mean, you can produce the Fibonacci numbers one after another, but we're going to derive a formula for the  $n$ -th Fibonacci number.

And when we're going to do it, we're going to do it more broadly for a class of linear recurrences. So let me define what that is. So a recurrence is said to be linear if it is of the form  $f$  of  $n$  equals a constant  $a_1$  times  $f$  of  $n$  minus 1 plus  $a_2$  times  $f$  of  $n$  minus 2 plus dot dot dot  $d$ 'th constant  $a_d$  times  $f$  of  $n$  minus  $d$ .

And we could simplify that as the sum  $i$  equals 1 to  $d$  of  $a_i$  times  $f$  of  $n$  minus  $i$ . And the constants are fixed here, so for fixed  $a_i$  and  $d$ . So the number of terms has to be a constant, and each coefficient has to be a constant. Can't vary.

And we define  $d$  to be the order of the recurrence. So  $d$  is the order of the recurrence. OK? And you can see, of course, that Fibonacci's recurrence is linear. What's its order? 2. And the coefficients, the  $a$ 's, are just 1. So it's a simple linear recurrence.

All right. So let's see how to solve it. Well, actually, before we do that, can you see the difference between this recurrence, linear, and divide-and-conquer recurrences? Right? What do I have here inside for a divide-and-conquer recurrence? I get a fraction of  $n$ , right?

And here, I'm subtracting a constant from  $n$ , usually like 1, 2, 3, an integer from  $n$ . So linear is

when inside you have  $n$  minus 1,  $n$  minus 2. Divide and conquer, you got  $n/2$  or  $3/4 n$ . And it makes a huge difference in the solution.

All right. So what I'm going to do is give you a closed-form solution for these kinds of recurrences. And it won't be completely easy because it took Europeans six centuries to find the solution to this. Right? Fibonacci discovers the thing in 1200 and tells everybody about it. And it wasn't until 600 years later that they figured out a closed-form solution.

So let's do that. Now, what we're going to do to do it this first time ourselves, and we don't have the formula, is to use guess and verify. So we're going to guess a solution and check that it works. And we're going to guess, really, a class of solutions. We're going to try  $f$  of  $n$  is an exponential in  $n$ ,  $\alpha$  to the  $n$  for some constant  $\alpha$ .

Now, we're going to figure out what  $\alpha$  is as we go along during the verification process. All right? So let's try to verify this guess and plug it in. We know that  $f$  of  $n$  is  $f$  of  $n$  minus 1 plus  $f$  of  $n$  minus 2. Well, let's plug that in and see what we get. That gives us  $\alpha$  to the  $n$  equals  $\alpha$  to the  $n$  minus 1 plus  $\alpha$  to the  $n$  minus 2. All right?

Now, that means I can divide by  $\alpha$  to the  $n$  minus 2, and I get  $\alpha$  squared equals  $\alpha$  plus 1. And now I can use the quadratic formula to solve for  $\alpha$ . All right? That means that  $\alpha$  squared minus  $\alpha$  minus 1 equals 0, which means that  $\alpha$  equals-- minus minus 1 is 1 plus or minus the square root of-- minus 1 squared is 1 minus  $4ac$  plus  $4$  over  $2$ . OK?

So there's two possible solutions here. This is 1 plus or minus the square root of 5 over 2. So it works if  $f$  of  $n$  equals either-- well, we'll call the roots here  $\alpha_1$  and  $\alpha_2$ .  $\alpha_1$  will be the positive case, 1 plus square root of 5 over 2. And  $\alpha_2$  will be the negative case, 1 minus square root of 5 over 2. All right? So guess and verify works so far if we have an exponential with either one of those bases. All right?

By the way, does anybody recognize that number? It's a famous number.

**AUDIENCE:** Golden ratio.

**PROFESSOR:** The golden ratio, which is supposed to have all these magical mystical properties. That when you look at a building, if its aspect ratio is that, it's perfect to the human eye. I don't know. But there's a lot of stuff about the golden ratio that happens to come up here. OK?

Now in fact, there's more than just these two solutions. It turns out that whenever you have a

linear recurrence and you've got two or more solutions like that, any linear combination is also a solution. All right? So we're just going to state that as a fact. It's not too hard to prove, but we won't prove it in class.

So if  $f$  of  $n$  equals  $\alpha_1$  to the  $n$ -- and this is true for any  $\alpha_1$  and  $\alpha_2$ -- and  $f$  of  $n$  equals  $\alpha_2$  to the  $n$  are solutions to a linear recurrence-- and here I mean without yet applying the boundary conditions. So far, we've ignored the boundary conditions, and we'll just do that for a little longer.

So if there are solutions without worrying about the boundary conditions, then  $f$  of  $n$  equals  $c_1$  times  $\alpha_1$  to the  $n$  plus  $c_2$   $\alpha_2$  to the  $n$  is also a solution for any constants  $c_1$  and  $c_2$ . All right? So any linear combination of our solutions also works. If you plugged it in to do verify, it would be fine.

So that means that  $f$  of  $n$  equals  $c_1$  times  $1$  plus square root of  $5$  over  $2$  to the  $n$  plus  $c_2$   $1$  minus square root of  $5$  over  $2$  to the  $n$  is a solution-- oops-- to Fibonacci's recurrence without boundary conditions again. All right? So I could plug this expression into the recurrence, and it would satisfy it. I won't do that.

But we haven't dealt with the boundary conditions yet. And in fact, dealing with the boundary conditions is what determines the values of these constants. I could have Fibonacci's recurrence where  $f$  of  $0$  was  $10$  and  $f$  of  $1$  was  $20$  if I wanted to, and then the recurrence would be the same afterwards. And it will turn out I get different constants here. All right? But otherwise, the form is going to look like this.

So let's see how to make that work. Let's see how to determine the constant factors. OK. So to determine the constant factors, we plug in the boundary conditions. So we have  $f$  of  $0$  equals  $0$  from the boundary condition. And now we plug that into our formula over there. That's  $c_1$  times  $\alpha_1$  to the  $0$  plus  $c_2$   $\alpha_2$  to the  $0$ . Of course, anything to the  $0$  is just  $1$ . And that means that  $c_2$  equals minus  $c_1$ . All right?

And now I'll use the next boundary condition to nail them down. All right? So I know also that  $f(1)$  is  $1$ , and that equals  $c_1$   $1$  plus square root of  $5$  over  $2$  to the first power plus  $c_2$   $1$  minus the square root of  $5$  over  $2$  to the first power. I plug in  $c_2$  as minus  $c_1$ . So I get  $c_1$   $1$  plus square root of  $5$  over  $2$  minus  $c_1$   $1$  minus square root of  $5$  over  $2$ .

And now that I can factor out the  $c_1$ , I get  $1$  minus  $1$  square root of  $5$  minus minus square root

of 5, which gives me  $c_1 = \frac{1 + \sqrt{5}}{2}$ . These cancel. This was all equal to 1, so that means that  $c_1$  equals  $\frac{1}{\sqrt{5}}$ . And of course,  $c_2$  is minus that.  $c_2$  is  $-\frac{1}{\sqrt{5}}$ . Any questions so far? All right.

Now I could write out the formula for the Fibonacci numbers. All right? So the solution is  $f_n$  equals  $c_1^n$ , which is  $\frac{1 + \sqrt{5}}{2}$  to the  $n$ -th power plus  $c_2^n$ , which is  $-\frac{1}{\sqrt{5}}$  to the  $n$ -th power. And that is the formula for the  $n$ -th Fibonacci number.

You wouldn't have guessed that to start with obviously. That would require pretty divine inspiration. And you can sort of see why it took them 600 years in Europe to figure out the answer. All right? It's not the first thing you'd think about. In fact, if somebody told you the answer and said this is it, you'd go, oh, give me a break. All right?

It does not look like-- I mean, what are the chances that evaluates to an integer? All right? It's got square root of 5's all over the place. Right? And here I'm telling you that  $f_6$ , the sixth Fibonacci number, which is 8, 3 plus 5, I'm telling you that is equal to  $\frac{1 + \sqrt{5}}{2}$  to the sixth power plus  $-\frac{1}{\sqrt{5}}$  to the sixth power.

I mean, would you believe me if I told you that? Probably not. What are the chances that's true? Somehow, magically, all those square root of 5's all go away, and you're just left with 8, a simple integer. OK? All right.

Yeah, there's sort of some more interesting things about this. What happens to this value here as  $n$  gets large? What does it do as  $n$  gets large?

**AUDIENCE:** Gets small.

**PROFESSOR:** It gets small Because  $\frac{1 - \sqrt{5}}{2}$  is about 0.6. This is about, here, this is about 0.618 something or other. And if I take a fraction less than 1 to the  $n$ -th power, it goes to 0. It goes away. So in fact, as  $n$  gets large, this is what the  $n$ -th Fibonacci number starts to look like.

In particular,  $f_n$  equals just that first one--  $\frac{1 + \sqrt{5}}{2}$  times the golden ratio to the  $n$ -th power plus some error term,  $\delta_n$ . And  $\delta_n$  is less than a  $\frac{1}{10}$  for  $n$  greater than 4. And it is little  $\delta$  of 1 in general going to 0. So in fact, the  $n$ -th Fibonacci number is about the  $n$ -th power of the golden ratio divided by square root of 5. Yeah?

**AUDIENCE:** [INAUDIBLE] top of the right board--

**PROFESSOR:** Yeah?

**AUDIENCE:** --you say that you can pick sort of any constants that satisfy it?

**PROFESSOR:** Yep.

**AUDIENCE:** Does that shift the sequence over, or what does that do? I guess I don't understand.

**PROFESSOR:** OK. You're asking, what are these things doing here?

**AUDIENCE:** Yeah. And why [? isn't there-- ?] why are they also solutions? Because this is the right ones?

**PROFESSOR:** The right one. OK. So there's two parts to a recurrence. There's this part, OK, and there's the boundary conditions--  $f(0)$  equals 0 and  $f(1)$  equals 1. There are really two parts to it. If you change the boundary conditions, you would change the rest of the terms, right? If I started with 10 and 20,  $f$  of 2 would be 30, OK, if I'm using this recurrence form for  $n$  bigger than or equal to 2.

Now, so when I'm computing a solution, first I'm saying, let's ignore the boundary conditions and look at all the recurrences that share this part. And that family of recurrences, I'm claiming, have these solutions. This is the solution space if I just worry about this part and not the boundary conditions.

Now, once I plug the boundary conditions in, that determines which values of  $c_1$  and  $c_2$  I get. And so if I take the version of the recurrence where  $f(0)$  is 0 and  $f(1)$  is 1, then  $c_1$  is going to be  $1$  over root 5, and  $c_2$  is going to be negative  $1$  over root 5.

If I use this recurrence with  $f(0)$  equals 10 and  $f(1)$  equals 20, I get different constants, but these powers stay the same. In fact, maybe we'll make an exercise like that from the problem set to figure out what it is if I started with 10 and 20. OK? So that's what the constants have to do-- they come from the boundary condition, which is sort of a cool fact. Does that make sense?

**AUDIENCE:** Yeah.

**PROFESSOR:** Any other questions about what I was doing? So we used guess and verify, but I sort of

guessed a form first. And as I verified, I sort of revised my guess along the way. All right? So I started guessing this. And then I said, well, let's refine that guess so that really it's one of these two guys.

And then I said-- I used a fact that I didn't prove-- that said I could use really any linear combination, plugged it into the base cases, and got my constants. All right? Now, this will lead up to a formula or an approach so you don't have to go through this on your own. You just plug in the formula in general.

OK. Let's get back to the original question of when all the jobs are filled. For what value of  $n$  is  $f$  of  $n$  bigger than  $M$ ? All right? Let's do that. OK. To see when all the jobs are filled, all  $M$  jobs, for the  $n$  when  $f$  of  $n$  is bigger and equal to  $M$ , when we got a Fibonacci number that's bigger than  $M$ .

And we just showed that basically  $f$  of  $n$ , well, it equals this where that's a tiny thing. All right? So we need to figure out when is  $1$  over square root of  $5$  times the golden ratio to the  $n$ -th power-- plus I have this tiny little thing that doesn't matter-- is bigger than or equal to  $M$ ? So I can solve for  $n$  now by subtracting the delta term, multiplying by square root of  $5$ . That gives me the golden ratio to the  $n$ -th power bigger than or equal to square root  $5$  times  $M$  minus the tiny thing.

Now I take logs. And I'm in trouble when  $n$  is bigger than or equal to  $\log$  of this over  $\log$  of the base. In other words, this is  $\theta$ . What's this  $\theta$  of here? That goes away. I can skip all the constant factors. What's this  $\theta$  of here, this expression, in terms of  $\theta$  of something to do with some function of  $M$ ?

**AUDIENCE:** Log of  $M$ ?

**PROFESSOR:** Log of  $M$ . So the jobs get filled in  $\log$  of  $M$  years. And in fact, if I plugged in, for example,  $M$  equals  $10,000$  into this expression, I would find that all the jobs are filled in  $20$  years, which is more or less what happened in computer science. Ballpark, those numbers are roughly correct. All right? But you can solve it exactly now by just plugging in whatever  $M$  you want. That's a fraction less than  $1/10$ . And you can solve to find out when your population gets that big. OK. Any questions?

OK. So now what I want to do is show you how to use this same idea to solve any linear recurrence, and this is the process you'll go through to do it. All right. So we're going to solve a

general linear recurrence. So in this case, we've got  $f$  of  $n$  is the sum from  $i$  equals 1 to  $d$ , for an order  $d$  recurrence, a sub  $i$   $f$  of  $n$  minus  $i$ .

And we've got to have boundary conditions. So we'll have  $f$  of 0 is  $b_0$ ,  $f$  of 1 is  $b_1$ . And how many boundary conditions do you think I'm going to need to have? Any guesses? I had two for Fibonacci. In this case, I've got  $d$  terms.

**AUDIENCE:**  $d$ .

**PROFESSOR:**  $d$ . I'm going to need  $d$  boundary conditions. So we'll go all the way to  $f$  of  $d$  minus 1 equal to  $b$  of  $d$  minus 1. All right? And now I'm going to try  $f$  of  $n$  is  $\alpha$  to the  $n$ . We're going to plug it into this expression. And then when we do, we get  $\alpha$  to the  $n$  equals  $a_1 \alpha$  to the  $n$  minus 1 plus  $a_2 \alpha$  to the  $n$  minus 2 all the way down to a sub  $d$   $\alpha$  to the  $n$  minus  $d$ . All right? Just plugging into there.

I can divide everything by  $\alpha$  to the  $n$  minus  $d$ , and that gives me  $\alpha$  to the  $d$  equals  $a_1 \alpha$  to the  $d$  minus 1 plus  $a_2 \alpha$  to the  $d$  minus 2 plus  $a_d$  times  $\alpha$  to the 0, which is just 1. And I can rewrite this as a polynomial equal to 0. So that means that  $\alpha$  to the  $d$  minus  $a_1 \alpha$  to the  $d$  minus 1 minus  $a_2 \alpha$  to the  $d$  minus 2 minus  $a_d$  is 0. This is called the characteristic equation of the recurrence. OK?

Now, what you need to do is compute the roots of this, and we'll use the roots to get the solution to the recurrence. Now, the simple case is when all the roots are different-- so let's do that first-- like there was in the Fibonacci example. All  $d$  roots are different, and let's call them  $\alpha_1$  to  $\alpha_d$ .

In that case, the solution is all linear combinations of their  $n$ -th powers, just like it was with Fibonacci without the boundary conditions.  $c_1 \alpha_1$  to the  $n$  plus  $c_2 \alpha_2$  to the  $n$  plus  $c_d \alpha_d$  to the  $n$ . All right? Same thing happens that happened in Fibonacci, that this becomes the solution before the boundary conditions are applied. All right?

And by the way, if your characteristic equation has imaginary roots, that's fine. It doesn't matter because the imaginary  $i$  term will disappear just like the square root of 5 disappeared. Has to because we know  $f$  of  $n$ , in this case, is a real number. All right? OK. Now, to find out the values of the coefficients, what are we going to do? What is it?

**AUDIENCE:** Boundary conditions.

**PROFESSOR:** Boundary conditions, yep. So now we solve for  $c_1, c_2, \dots, c_d$  from the boundary conditions. That is,  $f(i) = b(i)$  for  $i$  from 0 to  $d$ . So for example, we know that  $f(0) = c_1 \alpha^1 + \dots + c_d \alpha^d + c_0$ , which is 1,  $c_2 \alpha^2 + \dots + c_d \alpha^d + c_0$ . And that's going to be equal to  $b_0$  and so forth.

So you get a system of equations,  $d$  equations in  $d$  variables, that you solve to find the coefficients. And that will give the unique and correct solution to the recurrence. All right? Now, turns out this system of equations is never degenerate. It always has a solution. And you can prove that, but we won't do that. Any questions about how to do that? All right. Yeah?

**AUDIENCE:** Sorry, I just had a quick question. But what exactly was the distance there between  $n$  and  $d$ ?  $d$  is like the degree of-- which the terms you go back to.

**PROFESSOR:** Yes, Yeah.  $d$  is how far back you're going. You go as far back as  $f(n-d)$ .

**AUDIENCE:** OK. Cool.

**PROFESSOR:** And that becomes the degree, and that is a constant-- 2, 3, 4. Well, I don't think we'll ever ask you anything beyond 4 because it gets to be a pain to do. Typically, it's 2 or 3, sometimes even just 1. And then that becomes the power of your characteristic equation. It's the order of the characteristic equation. Any other questions?

All right. That was the nice case where all the roots of your characteristic equation were different. If they're not all different, it's a little more complicated. So the tricky case is repeated roots.

Now, the theorem, which we won't prove but tells you what to do, is that if  $\alpha$  is a root of the characteristic equation and it is repeated  $r$  times-- so  $(x - \alpha)^r$  is a factor-- then  $\alpha, n\alpha, n^2\alpha, \dots, n^{r-1}\alpha$  are all solutions to the recurrence.

All right? And then you would treat them just as you would the other roots. You take linear combinations, just like you did with the other roots to put it all together. By the way, is anybody starting to recognize a similarity with something else that you've studied in the past?

**AUDIENCE:** Differential equations.

**PROFESSOR:** Differential equations. This is the discrete analog of differential equations. All right? Recurrences is the same thing. All the math we're going to do henceforth is going to look just

like what you did with differential equations. So that's sort of good news and, I guess, bad news, depending on whether you like that stuff.

**AUDIENCE:** Bad news.

**PROFESSOR:** Bad news. Yeah, OK. All right, let's do an example that uses maybe the repeated roots case. Suppose there's a plant out there, and this plant lives forever but only reproduces in the first year of life and then never again. And it reproduces one for one. All right? Let's see what happens in that case.

Actually, there's a plant sort of like this in Hawaii. I think it's called the century plant, and it's rare. We'll see why when we solve this problem. So it reproduces one for one during the first year of life, then never again, and the plant lives forever. So our question is, how fast does the plant population grow? All right? In year  $n$ , how many plants there are.

So to figure that out, we're going to set up a recurrence. We're going to let  $f$  of  $n$  be the number of plants in year  $n$ . And we're going to say that the first plant miraculously comes into existence in year 1. So that in year 0 there's none of them, and in year 1 there's 1. And now we want to know how many there are in year  $n$ . OK. So let's set up the recurrence.  $f$  of  $n$  equals-- well, the previous year, how many plants were there in terms of  $f$ ?

**AUDIENCE:** [INAUDIBLE]

**PROFESSOR:**  $f$  of  $n$  minus 1. All right? That's how many there were last year, and they're all still alive, plus we've got to add the number of new plants. Well, that would be-- yeah, well, how many new plants are there? It's all the plants that were one-year-old last year.

So that's all the ones that were alive last year minus the ones that were alive the year before. That's how many new plants there were last year, and they each produce 1 new one this year. Is that OK? Everybody buy that? OK. So that equals  $2f$  of  $n$  minus 1 minus  $f$  of  $n$  minus 2. OK. What's my characteristic equation for this recurrence? Yeah?

**AUDIENCE:** Alpha squared minus 2 alpha plus 1.

**PROFESSOR:** Yes. OK? alpha squared minus 2 alpha plus 1. All right? We jumped to that answer. I could have written down alpha to the  $n$  minus 2 times alpha to the  $n$  minus 1 plus alpha to the  $n$  minus 2 and then divided by alpha to the  $n$  minus 2. But pretty quickly, you want to start just reading it off.

That's the characteristic equation. I get alpha squared, and then bringing this over, a minus 2 alpha, bring this over, plus 1 equals 0. Any questions on that? This is the process now you will always use that we're going through. All right. What are the roots to this thing? What are the roots to the characteristic equation?

**AUDIENCE:** Alpha minus 1 [INAUDIBLE].

**PROFESSOR:** Yeah. Alpha minus 1 times alpha minus 1 is 0, which means alpha equals?

**AUDIENCE:** 1.

**PROFESSOR:** 1. So the roots are alpha equals 1, and it's a double root. OK? It's a double root. It occurs twice because this is alpha minus 1 squared. OK. So what solutions am I going to use to my recurrence? What's f of n going to be now before I put in the boundary conditions? Well, it's going to be c1 times what to the n?

**AUDIENCE:** 1.

**PROFESSOR:** 1. That's the first root. What's here? What's the next thing I'm going to put here?

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** n times 1 to the n. All right? Because I've got a root that's repeated twice, r equals 2, so I have alpha to the n and n alpha to the n. Just happens alpha's 1, which makes it really easy. All right? So this is now my solution before the boundary conditions. In fact, that gets even simpler. That's 1. That's c1 plus c2 times n.

And now all that's left is to plug in the boundary conditions. So let's do that. Well, f of 0 equals 0. If I plug it into here, what happens for f of 0? What does that become?

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** C1. So c1 equals 0. That's good. f of 1 equals 1. And now that's going to be c1 plus c2. c1 is 0, so it means c2 equals 1. All right. So this is really easy. What's f of n equal? n. So we went through a lot of work to get an answer that probably we could have guessed pretty easily by just plugging in a few examples.

But the nice thing is this works for all the cases when it's not so easy to guess by plugging in the examples. Yeah, now you can see why this is a relatively rare plant. Its population is

growing very slowly. OK. Any questions about that? That had repeated roots. All right. Yeah?

**AUDIENCE:**

So we just guessed that  $f$  of  $n$  is equal to  $\alpha$  to the  $n$ , and we don't know if that works for like different [INAUDIBLE]. But how do you know that it works until you plug it in? If you tried it out [INAUDIBLE]?

**PROFESSOR:**

Yeah. We didn't go through and prove it. We used some facts along the way that we didn't prove, which you can do. Like, in particular, the fact we didn't prove that theorem there that those will be the roots. And we didn't prove that any linear combination would be a solution. But those are facts you could take from-- you could actually prove them. They're not too hard to prove.

But otherwise, I think we went through every step and narrowed it in. If we wanted to be really sure, we'd go back and prove it by induction against the original recurrence. And we'd say this would be the induction hypothesis. We'd cover the base cases--  $f$  of 0 and  $f$  of 1.

And then we'd plug this into the recurrence up there, which is, does  $n$  equal twice  $n$  minus 1 minus  $n$  minus 2, is what we would do. And you see that it's true. So you could really prove it by induction for any of the solutions we ever get. And if you worked this way to get the solution, it will always work. This method never fails, gives you the wrong answer.

In fact, that's an important point. When we tell you to solve a recurrence using this method on an exam or for homework, you don't have to verify it by induction unless we say verify by induction. Because we're giving you a method that is guaranteed to work. And if you do the method right, you're fine. There's no guessing here so far. In a minute, we're going to do a little guessing, but not so far. Any other questions? OK.

This works for all homogeneous linear recurrences. All right? Remember homogeneous and inhomogeneous from differential equations? It's the same thing happens here. And now we're going to talk about the inhomogeneous case.

All right. So we've been looking at linear recurrences, which means you have something like  $f$  of  $n$  minus  $a_1 f$  of  $n$  minus 1 minus  $a_d f$  of  $n$  minus  $d$  equals 0, and that means it's homogeneous. Now, instead of 0, I might have had something else here, might have been equal to 1, maybe  $n$  squared, or some general function  $g$  of  $n$ . These cases are all inhomogeneous. OK?

Now, solving inhomogeneous linear equations is just one step harder than homogeneous.

Yeah?

**AUDIENCE:** All right. Could you define homogeneous and inhomogeneous for people who didn't take differential equations?

**PROFESSOR:** Yeah. Homogeneous means you have the 0 here. You got all these  $f$  terms here, right, and there's nothing else.  $f$  of  $n$  minus  $a_1 f$  of  $n$  minus 1 dot dot dot minus  $a_d f$  of  $n$  minus  $d$  equals 0. Like with Fibonacci numbers--  $f$  of  $n$  minus  $f$  of  $n$  minus 1 minus  $f$  of  $n$  minus 2 is 0. OK?

I could consider other recurrences where there's something else out here that it equals. Like I could have a Fibonacci recurrence where I have  $f(n)$  equals  $f$  of  $n$  minus 1 plus  $f$  of  $n$  minus 2 plus  $g$  of  $n$ , like  $n$  cubed. That would give me a Fibonacci-like recurrence. And as soon as you put a non-zero out here, then it's inhomogeneous. And now I'm going to tell you how to solve that kind of recurrence, which is more general. OK? OK.

So let me outline the method, and then we'll do an example. All right. So the general inhomogeneous recurrence is exactly this. It's  $f$  of  $n$  minus  $a_1 f$  of  $n$  minus 1 minus  $a_d f$  of  $n$  minus  $d$  equals  $g$  of  $n$ , where that's some fixed function of  $n$ , nothing to do with  $f$ .

And we solve it according to the following three-step method. In step 1, we replace  $g$  of  $n$  by 0, thereby creating the situation we already know how to solve. And we solve the homogeneous recurrence-- because you got a 0 there now-- but we don't go all the way. We ignore the boundary conditions for now. So you don't get all the way to the final answer where you plugged in the boundary conditions to get the constant coefficients. Leave the constant coefficients undecided.

Then in step 2, we put back in  $g$  of  $n$ , and we find any what's called particular solution, again ignoring boundary conditions. So you'll have the constant factors there too. All right? And then step 3 is going to be to put the whole thing together and plug in the boundary conditions and solve it.

So we add the homogeneous and particular-- and I'll show you how to do step 2 in a minute-- so we add the homogeneous and particular solutions together and then use the boundary conditions to get the answer.

OK. So let's do an example. Let's go down here and do an example of the three steps. So let's say that our recurrence is this nasty-looking thing--  $f$  of  $n$  equals  $4 f$  of  $n$  minus 1 plus 3 to the

n. And the boundary condition is that  $f(1) = 1$ .

So step 1 says, ignore this  $3$  to the  $n$  thing and get back to just the homogeneous form, which is  $f(n) - 4f(n-1) = 0$ . So let's solve that. What's the characteristic equation for the homogeneous part? What's the characteristic equation? What is it?

**AUDIENCE:** Alpha [INAUDIBLE].

**PROFESSOR:** Close. It's-- you could either say  $\alpha$  to the  $n - 4\alpha$  to the  $n - 1$ . But better to simplify it into an order 1 polynomial. So it would be  $\alpha - 4$ , really simple in this case. So the characteristic equation is  $\alpha - 4$ ,  $\alpha - 4$ , all right, equals 0.

And it's really easy to find the root of this thing. It's just  $\alpha = 4$  is your only root. And that means the homogeneous solution is  $f(n) = \text{constant} \times 4^n$ . All right? So that's step 1.

OK. Let's do step 2, which we've not tried before. We need to find a particular solution, and this just means any old solution to  $f(n) - 4f(n-1) = 3^n$ , without worrying about boundary conditions.

Now, there's basic rules to use to figure out what to guess here. And basically, the idea is that you guess something for  $f(n)$  that looks a whole lot like this  $g$  term out here. And in particular, if this  $g$  term is  $3^n$ , you guess a constant times  $3^n$ . If it's  $5^n$ , you guess a constant times  $5^n$ . If it's  $n^2$ , you guess a polynomial that's of degree 2 and so forth. OK?

So let's guess a constant times  $3^n$  and see if we can make it work, not worrying about the boundary conditions. All right. And this is just like differential equations, right? Same guessing strategy exactly you use in differential equations, those of you who've had that.

So we guess  $f(n) = \text{constant} \times 3^n$ , and let's plug it in. So we plug that in up there. We get  $c3^n - c3^{n-1} = 3^n$ . All right? So let's divide by  $3^n$  here. We get  $3c - c = 3$ . Did I do that right? No, I left off my 4 here, right, left off that. So there's a 4 here. And that means that  $c = -3$ . OK? So I got  $c = -3$ , so  $c = -3$ .

All right. That means that the particular solution is just  $f(n) = -3^n$ . All right? So now I found a solution where there's no constants involved. This

time, the constant went away just plugging into the recurrence formula. I didn't use base cases yet. I just found a particular case that works for the recurrence-- minus 3 to the n plus 1. OK? All right.

Now we go to step 3. All right. Step 3, we find the general solution by adding the homogeneous and the particular solution. So we take  $f$  of  $n$  equals  $c_1$  times  $4$  to the  $n$  plus negative  $3$  to the  $n$  plus  $1$ . And all I've got to do is determine  $c_1$ , and I do that from the boundary conditions.

Yeah, let's see. Where was my boundary condition here? Ah, up there--  $f$  of  $1$  is  $1$ . So  $f$  of  $1$  equals  $1$ . Plugging in  $1$  here, I get  $c_1$  times  $4$  to the  $1$  minus  $3$  squared. All right? So I have  $4c_1$  minus  $9$ . Put the  $9$  over here, I get  $10$  equals  $4c_1$ . And that means that  $c_1$  is  $5/2$ . And now I know the final solution for  $f(n)$ . It's  $5/2$   $4$  to the  $n$  minus  $3$  to the  $n$  plus  $1$ .

Now, when you're all done with this, you don't have-- we won't make you do an inductive proof, which you could do to verify it's right. If you wanted to be really, really careful, you should do that. But it is a good idea just to check a couple values of  $n$  to make sure you didn't make a mistake because you might have. With all these calculations, you might have made a mistake.

So let's check, for example,  $f$  of  $2$ . So by the recurrence,  $f$  of  $2$  is  $4$  times  $f$  of  $1$ -- it's  $4$  times  $1$ -- plus  $3$  squared, which is  $9$ , is  $13$ . And let's just check when we plug in  $2$  here.  $f$  of  $2$  is  $5/2$  times  $16$  minus  $27$ .  $5$  times  $8$  is  $40$ , minus  $27$  is  $13$ . Just as a sanity check. Because there's a decent chance if you made a mistake, it'll catch it pretty quick with  $n$  equals  $2$  or  $3$ . All right? Just to make sure you got it right. Any questions about that?

So the tricky part here is guessing the particular solution. So let me give you the rules for that, just write those down, and maybe I'll do one last example. All right. So we're guessing a particular solution. So if  $g$ -- that's saying the non-homogeneous part-- is exponential, you should try guessing an exponential of the same type.

So for example, say that  $g$  is  $2$  to the  $n$  plus  $3$  to the  $n$ . What you should do is guess some constant  $a$  times  $2$  to the  $n$  plus some constant  $b$  times  $3$  to the  $n$ . Plug it in-- not to the boundary conditions. Plug it into the recurrence equation and solve for  $a$  and  $b$ . And it generally will work.

If  $g$  is polynomial, you should guess a polynomial of the same degree. So for example, say  $g$

of  $n$  is  $n^2 - 1$ . You should guess-- let's see-- guess  $an^2 + bn + c$ . Plug that in for  $f$  of  $n$ . All right? So when we're doing these guesses, we're guessing  $f$  of  $n$ . Same thing up here.

Now, say you mixed two together. Suppose you have an example where  $g$  of  $n$  equals  $2^n + n$ . What do you suppose you do in that case?

**AUDIENCE:** [INAUDIBLE] to guess [INAUDIBLE].

**PROFESSOR:** Yeah, guess each one separately, add them together. So you're going to guess  $f$  of  $n$  equals  $a \times 2^n + bn + c$ . All right? Because you take the guess for that plus the guess for that.

All right. Now, there's one last thing that can go wrong here. And that is you can try your guess, and it doesn't work. All right? So there's rules for that too. For example, if, say,  $g(n)$  is  $2^n$  and your guess of  $a \times 2^n$ , where  $a$  is a constant, fails. And we'll do an example in a minute where it fails. What you do then is you guess a polynomial times  $2^n$ . So you'd guess  $an^2 + bn + c \times 2^n$ .

And if that fails, your next guess would be  $an^3 + bn^2 + cn + d \times 2^n$ . And it won't happen, but if that failed, you'd guess a cubic times  $2^n$ . You keep pounding it with another factor of  $n$  in front if the guess fails. And that's true for anything at all you would be doing like this, and that'll work.

And I don't think I've ever encountered an example where you have to go very far to make that fly. Same thing we had for repeated roots in the characteristic equation. Multiply those factors of  $n$  in front of it to get to the answer.

All right. Let's do one more example where it fails, so we get to see what happens. Let's try this recurrence--  $f$  of  $n$  is  $2f$  of  $n - 1 + 2^n$ . And the boundary condition is  $f$  of  $0$  is  $1$ .

What's the first thing I do for this?

**AUDIENCE:** Set it equal to  $0$ .

**PROFESSOR:** Set it equal to  $0$  and solve it. Get the homogeneous solution. OK. So what's the characteristic polynomial?  $\alpha - 2 = 0$ . What's my root?  $2$ . And therefore, my homogeneous solution

is  $c_1$  times  $2$  to the  $n$ . Well, that's pretty simple.

What's the next step? What do I have to find next?

**AUDIENCE:** Particular.

**PROFESSOR:** Particular solution. All right. So what am I going to guess?  $a$  times  $2$  to the  $n$ . All right? So let's plug that in,  $a$  times  $2$  to the  $n$ . I'm going to plug it in to here. So I get  $a$  times  $2$  to the  $n$  equals  $2$  times  $a$  times  $2$  to the  $n$  minus  $1$  plus  $2$  to the  $n$ . Did I do that right? Think so. So I get  $2$  to the  $n$ ,  $2$  to the  $n$ ,  $2$  to the  $n$ . I get  $a$  equals  $a$  plus  $1$ . Not so good. All right? There's no solution for  $a$ . Well, that's bad.

All right. So what do I do? Any thoughts about what I'm going to do next?

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** What is it?

**AUDIENCE:** Change your guess.

**PROFESSOR:** Change the guess. What's the next guess going to be? Yeah, all right. So now I'm going to guess  $f$  of  $n$  is  $a$  plus  $b$  times  $2$  to the  $n$ . We'll hope for better luck. So let's plug that in.  $a$  plus  $b$  times  $2$  to the  $n$  into-- where am I, up there-- equals  $2$  times  $a$  plus  $b$  times  $2$  to the  $n$  minus  $1$  plus  $2$  to the  $n$ .

All right. Now, I can divide out the  $2$  to the  $n$ 's here like I did before. And I get  $a$  plus  $b$  equals  $a$  minus  $a$  plus  $b$  plus  $1$ . That cancels here,  $b$  cancels. So I got  $a$  solution--  $a$  equals  $1$ . And I don't care what  $b$  is. I didn't need to set it to anything. So I'll make it  $0$  just to get it out of the way because it didn't matter.

So my particular solution then is  $f$  of  $n$ --  $a$  is  $1$ .  $b$  I can just make  $0$  because it didn't matter what I used, so I'll make it simpler. So I get  $n$  times  $2$  to the  $n$ . That's my particular solution. And I've got my general solution up there as  $c_1$  times  $2$  to the  $n$ -- sorry, the homogeneous solution.

What's the next step? What's step 3? What do I do with these guys, this solution and that solution?

**[? AUDIENCE:** Add them. ?]

**PROFESSOR:** Add them together. Good. So the general solution is I have  $f$  of  $n$  is the sum of  $c_1$  times  $2$  to the  $n$

plus the particular solutions  $n^2$  to the  $n$ . How do I figure out what  $c_1$  is?

**AUDIENCE:** Plugging in to [INAUDIBLE].

**PROFESSOR:** Plugging in the boundary condition--  $f(0)$  equals 1.  $f(0)$  equals 1. Plug in 0, I get  $c_1 2^0$  plus 0. Well, that's pretty easy.  $c_1$  equals 1. So I now have the final answer.  $f$  of  $n$  equals  $2^n$  plus  $n^2$  to the  $n$ . All right? Any questions? OK. So it's a little tedious to do this. But the really nice thing is any linear recurrence you ever see, this method always works, which is handy. OK. That's it for today.