

Final

- This final is **closed book**, but you may have three 8.5×11 " sheet with notes in your own handwriting on both sides.
- Calculators are not allowed.
- You may assume all of the results presented in class.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.
- For this final, \mathbb{N} is the set of nonnegative integers (including 0): $\mathbb{N} = \{0, 1, \dots, \}$.
- GOOD LUCK!
- **Important:** If you show your reasoning, even if your answer is wrong, you could earn partial credit.

Problem	Points	Grade	Grader
1	8		
2	20		
3	8		
4	10		
5	10		
6	28		
7	16		
Total	100		

Problem 1. [8 points] Prove that for all $n \in \mathbb{N}$, the following identity holds

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Problem 2. [20 points] Coin-Flip is a 2 player game. Each player wins with probability exactly 0.5. There are no ties.

n people are playing a Coin-Flip tournament. Every person plays a Coin-Flip game with every other person exactly once. Thus everybody plays $n - 1$ games. The outcomes of all the games are mutually independent of one another.

We say that the tournament is a *success* if for every $i \in \{0, 1, \dots, n - 1\}$, there is exactly one player, which we will refer to as p_i , with exactly i wins.

- (a) [10 points] Prove that if the tournament is a success, then for any integers j, k with $0 \leq k < j \leq n - 1$, p_j defeats p_k .
- (b) [6 points] What is the probability that the tournament will be a success?
- (c) [4 points] Show that your answer to part (b) is $o(1)$.

Problem 3. [8 points] A person is passing time by advancing a token on the set of natural numbers. In the beginning, a token is placed on 0.

The person keeps playing *moves* forever. Each move proceeds as follows:

1. First the person tosses a fair coin (with heads/tails equally likely).
2. Suppose the token is currently placed on n . If heads came up, then the person moves the token to $n + 3$, otherwise he moves the token to $n + 4$.

For each $n \in \mathbb{N}$, let E_n be the event "There was a move on which the token landed on n ". Let $p_n = \Pr[E_n]$.

Find a recurrence relation for p_n . *You do not need to solve the recurrence, but you should specify the boundary conditions that would be necessary to find a solution to the recurrence.*

Problem 4. [10 points] Exactly $1/5$ th of the people in a town have Beaver Fever[®].

There are two tests for Beaver Fever, TEST1 and TEST2. When a person goes to a doctor to test for Beaver Fever, with probability $2/3$ the doctor conducts TEST1 on him and with probability $1/3$ the doctor conducts TEST2 on him.

When TEST1 is done on a person, the outcome is as follows:

- If the person has the disease, the result is positive with probability $3/4$.
- If the person does not have the disease, the result is positive with probability $1/4$.

When TEST2 is done on a person, the outcome is as follows:

- If the person has the disease, the result is positive with probability 1.
- If the person does not have the disease, the result is positive with probability $1/2$.

A person is picked uniformly at random from the town and is sent to a doctor to test for Beaver Fever. The result comes out positive. What is the probability that the person has the disease?

Problem 5. [10 points] Two identical complete decks of cards, each with 52 cards, have been mixed together. A hand of 5 cards is picked uniformly at random from amongst all subsets of exactly 5 cards.

(a) [5 points] What is the probability that the hand has no identical cards (i.e., cards with the same suit and value. For example, the hand $\langle Q\heartsuit, 5\spadesuit, 6\spadesuit, 8\clubsuit, Q\heartsuit \rangle$ has identical cards.)?

(b) [5 points] What is the probability that the hand has exactly one pair of identical cards?

Problem 6. [28 points] Scores for a final exam are given by picking an integer uniformly at random from the set $\{50, 51, \dots, 97, 98\}$. The scores of all 128 students in the class are assigned in this manner. For parts (a), (b), (c) and (d) you may NOT assume that these scores are assigned independently. For parts (e), (f), (g) and (h) you MAY assume that these scores are assigned independently.

Let S_1, \dots, S_{128} be their scores. Let $S = \frac{1}{128}(\sum_{i=1}^{128} S_i)$ be the average score of the class.

(a) [3 points] For $i \in \{1, \dots, 128\}$, what is $\mathbb{E}[S_i]$?

(b) [2 points] Show that $\mathbb{E}[S] = 74$. Make no independence assumptions.

(c) [4 points] Prove that

$$\Pr[S \geq 88] \leq \frac{37}{44}.$$

Make no independence assumptions.

(d) [5 points] Improve your previous bound by using the fact that the minimum possible score is 50. Prove that

$$\Pr[S \geq 88] \leq \frac{12}{19}.$$

Make no independence assumptions.

(e) [4 points] For the remaining problems, assume that all the scores are assigned mutually independently. Use Problem 1 of this final to find $\text{Var}[S_i]$.

(f) [3 points] What is $Var[S]$?

(g) [2 points] What is the standard deviation of S ?

(h) [5 points] Prove, using the Chebyshev Inequality, that

$$\Pr[S \leq 69] \leq \frac{1}{16}.$$

Problem 7. [16 points] 1000 files $F_1, F_2, \dots, F_{1000}$ have just reached a disk manager for writing onto disk. Each file's size is between $0MB$ and $1MB$. The sum of all files' sizes is $400MB$.

The disk manager has 4 disks under its control. For each file F_i , the disk manager chooses a disk uniformly at random from amongst the 4 disks, and F_i is written to that disk. The choices of disk for the different files are mutually independent.

(a) [2 points] What is the expected number of files that will be written to the first disk?

We can use indicator variables. For each file, $P_i = 1$ if F_i is written to the first disk. The chance of an individual file being written to the first disk is $1/4$. By linearity of expectation, the expected number of files written to the first disk is the sum of the expected values of P_i 's. The expected value of each indicator variable is $1/4$, and $\sum_{i=1}^{1000} 1000)1/4 = 250$, so the expected number of files to be written to the first disk is 250.

(b) [2 points] What is the expected number of bytes written on the first disk?

We can say that each file F_i has bit size S_i . Each file has a $1/4$ chance of being written to the first disk. Therefore, by linearity of expectation, the expected number of bytes written to the first disk is the sum of the expected number of bytes per file written to the first disk, which is:

$$\sum_{i=1}^{1000} 1/4 \cdot S_i = 1/4 \sum_{i=1}^{1000} S_i = 1/4 \cdot 400 = 100$$

(c) [8 points] Find the best upper bound you can on the probability that $200MB$ or more are written on the first disk?

For this we can use the first Chernoff bound, which is:

$$\Pr(X \geq c \operatorname{Ex}(X)) \leq e^{-(c \ln c - c + 1) \operatorname{Ex}(X)}$$

The Chernoff bound only works if X is the sum of random variables that each take on a value between 0 and 1. The file size of each file in the first disk is between 0 and 1Mb. So we can define X to be the total number of bytes in disk 1. The expected value of X is 100, so we take c to be 2. We get:

$$\Pr(X \geq 2 \cdot 100) \leq e^{-(2 \ln 2 - 2 + 1)100}$$

(d) [4 points] Find the best upper bound you can on the probability that there is some disk with $200MB$ or more written on it?

For this we can use the Union Bound along with our result from above. The probability of this event happening in one or more disks is upper bounded by the sum of the probabilities of the event happening in each disk. This gives us an upper bound of

$$4 \cdot e^{-(2 \ln 2 - 1)100}$$

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