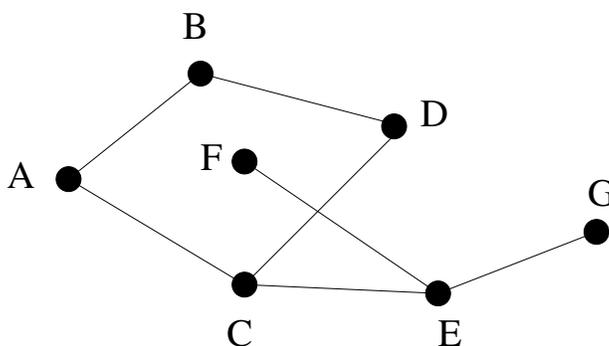


## Notes for Recitation 6

### 1 Graph Basics

Let  $G = (V, E)$  be a graph. Here is a picture of a graph.



Recall that the elements of  $V$  are called vertices, and those of  $E$  are called edges. In this example the vertices are  $\{A, B, C, D, E, F, G\}$  and the edges are

$$\{A-B, B-D, C-D, A-C, E-F, C-E, E-G\}.$$

Deleting some vertices or edges from a graph leaves a *subgraph*. Formally, a subgraph of  $G = (V, E)$  is a graph  $G' = (V', E')$  where  $V'$  is a nonempty subset of  $V$  and  $E'$  is a subset of  $E$ . Since a subgraph is itself a graph, the endpoints of every edge in  $E'$  must be vertices in  $V'$ . For example,  $V' = \{A, B, C, D\}$  and  $E' = \{A-B, B-D, C-D, A-C\}$  forms a subgraph of  $G$ .

In the special case where we only remove edges incident to removed nodes, we say that  $G'$  is the *subgraph induced on  $V'$*  if  $E' = \{(x-y | x, y \in V' \text{ and } x-y \in E)\}$ . In other words, we keep all edges unless they are incident to a node not in  $V'$ . For instance, for a new set of vertices  $V' = \{A, B, C, D\}$ , the induced subgraph  $G'$  has the set of edges  $E' = \{A-B, B-D, C-D, A-C\}$ .

### 2 Problem 1

An undirected graph  $G$  has *width*  $w$  if the vertices can be arranged in a sequence

$$v_1, v_2, v_3, \dots, v_n$$

such that each vertex  $v_i$  is joined by an edge to at most  $w$  preceding vertices. (Vertex  $v_j$  precedes  $v_i$  if  $j < i$ .) Use induction to prove that every graph with width at most  $w$  is  $(w + 1)$ -colorable.

(Recall that a graph is  $k$ -colorable iff every vertex can be assigned one of  $k$  colors so that adjacent vertices get different colors.)

**Solution.** We use induction on  $n$ , the number of vertices. Let  $P(n)$  be the proposition that every graph with width  $w$  is  $(w + 1)$  colorable.

*Base case:* Every graph with  $n = 1$  vertex has width 0 and is  $0 + 1 = 1$  colorable. Therefore,  $P(1)$  is true.

*Inductive step:* Now we assume  $P(n)$  in order to prove  $P(n + 1)$ . Let  $G$  be an  $(n + 1)$ -vertex graph with width  $w$ . This means that the vertices can be arranged in a sequence

$$v_1, v_2, v_3, \dots, v_n, v_{n+1}$$

such that each vertex  $v_i$  is connected to at most  $w$  preceding vertices. Removing vertex  $v_{n+1}$  and all incident edges gives a graph  $G'$  with  $n$  vertices and width at most  $w$ . (If original sequence is retained, then removing  $v_{n+1}$  does not increase the number of edges from a vertex  $v_i$  to a preceding vertex.) Thus,  $G'$  is  $(w + 1)$ -colorable by the assumption  $P(n)$ . Now replace vertex  $v_{n+1}$  and its incident edges. Since  $v_{n+1}$  is joined by an edge to at most  $w$  preceding vertices, we can color  $v_{n+1}$  differently from all of these. Therefore,  $P(n + 1)$  is true.

The theorem follows by the principle of induction. ■

### 3 Problem 2

A **planar graph** is a graph that can be drawn without any edges crossing.

1. First, show that any subgraph of a planar graph is planar.

**Solution.** The edge set of any subgraph will be a subset of the set of edges in the original planar graph. This means that since edges in the original graphs do not cross, edges in a subset of the original set of edges also do not cross. ■

2. Also, any planar graph has a node of degree at most 5. Now, prove by induction that any graph can be colored in at most 6 colors.

**Solution.** We prove by induction. First, let  $n$  be the number of nodes in the graph. Then define

$$P(n) = \text{Any planar graph with } n \text{ nodes is 6-colorable.}$$

*Base case,  $P(1)$ :* Every graph with  $n = 1$  vertex is 6-colorable. Clearly true since it's actually 1-colorable.

*Inductive step,  $P(n) \rightarrow P(n + 1)$ :* Take a graph  $G$  with  $n + 1$  nodes. Then take a node  $v$  with degree at most 5 (which we know exists because we know any planar graph has a node of degree  $\leq 5$ ), and remove it. We know that the induced subgraph  $G'$  formed in this way has  $n$  nodes, so by our inductive hypothesis,  $G'$  is 6-colorable. But  $v$  is adjacent to at most 5 other nodes, which can have at most 5 different colors between them. We then choose  $v$  to have an unused color (from the 6 colors), and as we have constructed a 6-coloring for  $G$ , we are done with the inductive step.

Because we have shown the base case and the inductive step, we have proved

$$\forall n \in \mathbb{Z}_+ : P(n)$$

(Note:  $\mathbb{Z}_+$  refers to the set of positive integers.) ■

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